APC SUMMARY LOCAL GRAPH ALG

$G=(V,E)$ $|V|=n$ traveler D → solve in $O(D)$ rounds

8.2 COLORING ROOTED TREE

$p(v)$ parent $\phi: V \to \{1 \dots q\}$ coloring $\phi(v) \neq \phi(p(v))$

trees → 2-coloring $\Omega(n)$
3-coloring

$\log^* x = \begin{cases} 0 & x \leq 1 \\ 1+\log^*\log x & \text{else} \end{cases}$

Thm 8.4 3-coloring for T in $\log^* n + O(1)$

Lem 8.5 k-col ϕ_{old}, $k \geq C_0$, compute k'-col ϕ_{new}
$k' = \log k + \log\log k/2 + 1$

$\Pi: \{1..k\} \to F_{k'}$ (set of all subsets of size $k'/2$) 1:1 map

$\phi_{new}(v) \in \Pi(\phi_{old}(v)) \setminus \Pi(\phi_{old}(u))$

$\binom{k'}{k'/2} \geq k$ Sperner family

Lem 8.6 k-col ϕ_{old}, $k \geq 4$ → 2 rounds → $(k-1)$-col ϕ_{new}

1. shift down
2. remove color k ($\{1,2,3\} \setminus \{\phi'_{old}(v), \phi_{old}(v)\}$)

$\phi_{new}(v)$ — $\phi'_{old}(v)$ par, $\phi_{old}(v)$ child

Thm 8.7 det alg 3-col needs $\left(\frac{\log^* n}{2} - 2\right)$ rounds

B k-ary q-col $\Leftrightarrow$ $B(a_1 \dots a_k) \in \{1..q\}$
$\neq B(a_2 \dots a_{k+1})$

Lem 8.10 There is no 1-ary q-col with $q < n$

Lem 8.11 If $\exists$ k-ary q-col B $\Rightarrow$ $\exists$ $(k-1)$-ary 2^q-col B'

8.3 DETERMINISTIC COLORING

Thm 8.12 Linial's coloring alg: deterministic, distributed
color n-node graph G with max deg $=\Delta$ using
$O(\Delta^2)$ colors, $O(\log^* n)$ steps

Def 8.13 $S_1, S_2, \dots S_k \subseteq \{1..k'\}$ ← ground set is Δ-cover-free family if
$\forall i_0, \dots i_\Delta \in \{1..k\}$ $S_{i_0} \setminus (\bigcup_{j=1}^{\Delta} S_{i_j}) \neq \emptyset$
(no set subset of union of all others)
(has at least 1 rep. elem)

Lem 8.14 $\forall k, \Delta$ $\exists$ Δ-cover free fam of size k
over $k' = O(\Delta^2 \log k)$

Lem 8.15 $\forall k, \Delta \geq k^{1/3}$ $\exists$ Δ-cover-free fam of size k
over $k' = O(\Delta^2)$
(or $k' = O(\Delta^2 \log_{k'}^2 k)$)

Lem 8.16 give ϕ_{old} k-col, in 1 round we compute
ϕ_{new} k'-col, $k' = O(\Delta^2 \log k)$.

if $k \leq \Delta^3$, $k' = O(\Delta^2)$

Kuhn-Wattenhofer

Lem 8.17 given ϕ_{old} k-col, $k \geq \Delta+2$, in 1 round
we compute ϕ_{new} $(k-1)$-col // trivial exclusion

Thm 8.18 $\exists$ det distr local alg, coloring G with
$(\Delta+1)$-col, $O(\Delta^2 + \log^* n)$ rounds

Lem 8.19 ϕ_{old} k-col, $k \geq \Delta+2$, compute ϕ_{new} $(\Delta+1)$-col
$O(\Delta \lceil \log(\frac{k}{\Delta+1}) \rceil)$

Thm 8.20 improve Thm 8.18 with $O(\Delta \log\Delta + \log^* n)$

Defective coloring

Thm 8.21 improve to $O(\Delta + \log^* n)$

Def 8.22 color assignment $\phi: V \to \{1..k\}$ is d-defective
if $\forall$ color $q \in \{1..k\}$ subgraph induced by
vertices of color q has max deg d (at most d
neighb with same col) → standard is 0-defective

Lem 8.23 d-defective k-col ϕ_{old}, 1 round
d'-def k'-col with $k' = O\left(\left(\frac{\Delta-d}{d'-d+1}\right)^2 \log k\right)$

8.4 NETWORK DECOMP

Def 8.24 Weak Diameter Network Decomp
$G=(V,E)$, (C,D) WDND is partition in disjoint
$G_1 \dots G_C$ st $\forall i \in \{1..c\}$ G_i made of clusters
$X_1 \dots X_\ell$ (v-disj mutually non adj), $v, u \in X_j$
have at most distance D in G

Def 8.25 Strong DND
each G_i has diameter $\leq D$ in G

Thm 8.26 with (C,D) WDND, we compute $(\Delta+1)$-col in
$O(CD)$ rounds

Thm 8.27 $\exists$ randomized bc alg, computes (C,D) WDND
$C = O(\log n)$, $D = O(\log n)$ in $O(\log^2 n)$ with high prob

randomized

Lem 8.28 max cluster diam $\leq O(\log n/\varepsilon)$ with high prob
→ can be computed in $O(\log n/\varepsilon)$

Lem 8.29 $\forall v$, prob (v not clustered), v on boundary and thus
gets discarded - is at most ε

Cor 8.30 after $C = O(\log_{1/\varepsilon} n)$ all nodes clustered
with high prob.

deterministic

Thm 8.31 ∃ det loc alg computing (C,D) strong SDND $C = 2^{O(\sqrt{\log n})} = D$ in $2^{O(\sqrt{\log n})}$ rounds

Thm 8.32 " $C = 2^{O(\sqrt{\log n \log\log n})} = D$ = # rounds

Def 8.33 $G=(V,E)$ $W \subseteq V$, (α,β)-ruling set of W in G is subset $S \subseteq W$ st
- $\forall u,v \in S$ $dist_G(u,v) \geq \alpha$
- $\forall v \in W$ $\exists u \in S$ st $dist(v,u) \leq \beta$

Lem 8.34 $G=(V,E)$, $W \subseteq V$ ∃ det loc alg computing $(k, k\log n)$-ruling set in $O(k \log n)$ rounds

Lem 8.35 each lvl i bundle has at least $(\Delta+1)^{i-1}$ vertices therefore $\log_\Delta n$ levels of iteration suffice

Lem 8.36 each lvl i bundle has diameter $O(\log n)^i$.
alg for lvl i can be performed in $O(\log n)^i \cdot O(\Delta + \log^* n)$ rounds

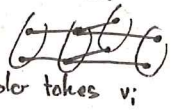
Cor 8.37 ... alg takes at most $\Delta \cdot O(\log n)^{\log_\Delta n}$ rounds
at end, we got SDND, C, D.
$\Delta = \ldots$

8.5 MAXIMAL INDEPENDENT SET

Def 8.38 $S \subseteq V$ is MIS if
- S independent, no two $v,v \in S$ are adjacent
- S maximal, $\forall v \notin S$ ∃ neighb v of v st $v \in S$

Lem 8.39 given loc alg A computing MIS in $T(\Delta)$ rounds, ∃ loc alg B computing $(\Delta+1)$-col in $T(n(\Delta+1))$

proof: $H = G \times K_{\Delta+1}$ $n(\Delta+1)$, $v_1 \ldots v_{\Delta+1}$



color takes v_i

Luby's alg $O(\log n)$

- each v picks $r_v \in [0,1]$, send to $N(v)$. Join S if it's bcs max ($r_v > r_u$ $\forall u \in N(v)$)
- if v joined, inform neighbours, all get removed

Thm 8.40 computes MIS $O(\log n)$ with high prob

$4\log n$ iterations $\rightarrow m_0/2^{4\log n} \leq 1/n^2$

$E[m_{i+1} \mid G_i$ has m_i edges$] \leq m_i/2$

"killed" ⊙ double counting

single-handedly kills v if r_w is max among $N(w) \cup N(v)$
→ limit double count to 2

$E[m_i - m_{i+1}] \geq \sum_{\{w,v\} \in E_i} \left(\frac{d(v)}{d(w)+d(v)} + \frac{d(w)}{d(w)+d(v)} \right)/2 = m_i/2$

↓ not relevant

8.6 SUBLOG-TIME RAND COL

Thm 8.41 ∃ rand loc alg computing $\Delta(1+\varepsilon)$-col in $O(\sqrt{\log n})$ rounds $\varepsilon > 0$ with high prob.

low deg

Thm 8.42 ∃ rand alg " $\Theta(\sqrt{\log n}/\varepsilon)$ assuming $\Delta \geq 2^{\sqrt{\log n}}$

Lem 8.43 after $\Theta(\sqrt{\log n}/\varepsilon)$, with high prob each connected comp in remaining subgraph has diameter $\leq \Theta(\sqrt{\log n}/\varepsilon)$

proof $(1-\varepsilon)^k \leq \exp(-\varepsilon k) \leq (-C^2 \log n/\varepsilon)$
$n\Delta^\ell \cdot \exp(-C\log n) = \ldots \leq 1/n^c$

high deg

Thm 8.44 ∃ rand alg computing $\Delta(1+\varepsilon)$-col, $\varepsilon > 0$ in $\Theta(\sqrt{\log n}/\varepsilon)$ rounds

Lem 8.45 with high prob, each subgraph G_i has max deg $\leq \frac{\Delta}{k}(1+\varepsilon/3)$

Thm 8.46 Chernoff bound
$X_1 \ldots X_\eta$ indep indicator var. $X = \sum X_i$, $\mu = E[X]$
$\forall \delta > 0$ $Pr[X \notin [\mu(1-\delta), \mu(1+\delta)]] \leq 2e^{-\delta^2 \mu/3}$
(within const error of expected val is exponentially small)

$G \rightarrow G_i$: m_i $E[m_i] = \frac{m}{2^i} = \frac{m}{2^{4\log n}} = \frac{n^2}{n^4} = \frac{1}{n^2}$